Vortical Mode Instability of Shear Layers with Temperature and Density Gradients

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The linear inviscid instability of a low Mach number shear layer with imposed temperature and density gradients is analyzed in terms of its vortical, acoustic, and entropy modes. The conditions under which the acoustic mode is decoupled from both the vortical and entropy modes in the linear instability region of the shear layer are first discussed. It is then shown that for a two-dimensional parallel shear layer, satisfying these conditions, the vortical mode also decouples from the entropy mode. The instability of such a shear layer is then identified with the instability of the vortical mode. Two examples that illustrate the strong influence of the steady-state temperature profile on the growth rates of the unstable vortical mode are presented and discussed. Finally, it is demonstrated that the initial value of the unstable vortical mode is determined by the acoustic mode.

Nomenclature

A_1 = eigenfunction in Eq. (12)
a = local speed of sound
a_r = reference speed of sound
B' = Bernoulli enthalpy fluctuation
B = Fourier component of B'
C = constant in the solution of the unstable vortical wave
C_p = specific heat at constant pressure
c_{ph} = phase velocity of vortical wave
F = Fourier transform of right-hand side of Eq. (6)
G = Green's function
h = enthalpy
i = square root of -1
$\mathbf{i}$ = unit vector in axial direction
k = acoustic wave number
$\mathbf{k}$ = unit vector in direction perpendicular to the x-y plane
L = length of duct
L_1 = axial location of shear layer origin in duct
M = local Mach number
M_r = reference Mach number
m' = unsteady mass source
$\bar{p}$ = steady-state pressure
R_1, R_2 = parameters in boundary conditions for the acoustic motions
r' = vorticity fluctuation
r = Fourier component of r'
S = Fourier component of s'
s = entropy
$\bar{s}$ = steady-state entropy
s' = entropy fluctuation
$\bar{T}$ = steady-state temperature
T' = temperature fluctuation
t = time
$\mathbf{u}'$ = vortical component of velocity fluctuation
u_1', u_2' = vortical velocity fluctuations in x and y directions
u_1, u_2 = Fourier components of u_1' and u_2'
$\bar{u}$ = steady-state axial velocity
$\bar{u}_m$ = maximum value of $\bar{u}$
$\mathbf{v}$ = velocity vector
$\mathbf{v}'$ = velocity fluctuation vector
x, x_0 = axial coordinate
$\mathbf{x}$ = position vector
$\bar{x}$ = typical value of x
y, y_0 = transverse coordinate
y_c = transverse coordinate of generalized inflection point
α = wave number of unstable vortical wave
Θ = momentum thickness of the shear layer
λ = acoustic wavelength
ϕ = acoustic potential
Φ = Fourier component of ϕ
$\hat{\Phi}$ = amplitude of Φ
ψ = stream function for vortical velocity fluctuations
ψ_1 = stream function component for the homogenous problem
ψ_2 = stream function component for the acoustically forced problem
$\hat{\psi}_2$ = Fourier component of ψ_2
ω = circular frequency

Superscript
$*$ = nondimensional quantity

Introduction

SHEAR-layer stability and the transition from a laminar flow to a turbulent flow remain topics of considerable practical interest. For example, it is well known that the drag on and heat transfer to a body immersed in a flow are strongly dependent on whether the boundary layer over the body is laminar or turbulent.[1] Consequently, the attainment of a capability to predict the transition is of great practical importance. Also, interest in the study of shear-layer instabilities has increased in recent years when the importance of coherent turbulent flow structures in such problems as ramjet and afterburner combustion instabilities[2] and aircraft drag mechanisms[3] has become apparent. As these structures involve time and length scales consistent with those exhibited by the unstable disturbances in shear-layer instability studies, there is reason to believe that the instability of the shear layer plays a major role in the formation of these coherent structures.

These observed structures are characterized by a coherent distribution of unsteady vorticity and their behavior can dominate important features of shear layers such as mixing and spread. Thus, by controlling the dynamics of the coherent structures, the dynamics of the shear layer may also be controlled. An important technique utilized in this regard is acoustic excitation of the shear layer.[4,5] It has been found that

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such excitation can substantially affect the onset of the shear-layer instability as well as the characteristics of the ensuing turbulent flow. In this connection it needs to be pointed out that it is well known that acoustic motions are irrotational and, therefore, vorticity free. On the other hand, as already noted, the coherent flow structures are characterized by unsteady vorticity. Hence, acoustic excitation of shear layers represents a problem in which a solenoidal field (vorticity) interacts with an irrotational field (acoustics).

The relationship between vorticity and acoustics has been studied from the viewpoint of turbulence-generated sound for over 30 years. Lighthill[6] was the first to formulate a theory that predicts the sound generated from a region of turbulent flow. Since then, several other theories of aeroacoustics have been developed. In general, they invoke the assumption that the generated sound does not influence the turbulent flow dynamics; that is, the generated sound is a byproduct of the flow (Lighthill's hypothesis). From the viewpoint of sound generation, this is reasonable because turbulence is an inefficient quadropole source of sound and the magnitude of the generated acoustic velocities is much smaller than the turbulent flow velocities (that is, the "vortical" velocities). However, in the initial region of shear-layer formation the magnitude of the unsteady vortical velocities could be of the same order as the acoustic velocities so that the interaction between acoustics and vorticity in this region can influence the shear-layer development. In fact, low level acoustic forcing in shear layer formation regions has been used[7] to alter the downstream characteristics of the shear layer.

Hence, the influence of sound upon shear-layer instability and development is an area of considerable interest. Several papers dealing with this problem have appeared in the literature[8,9] in recent years.

These analyses have been confined to constant temperature and density shear layers. However, many practical shear layers have spatially varying temperature and density profiles. For example, the shear layer formed at the dump plane in a ramjet combustor has temperature and density gradients imposed by the heat addition from chemical reactions.[10] The present paper is concerned with similar shear layers having temperature and density gradients. It is assumed herein that the imposed temperature gradient is much larger than the temperature gradient caused by large flow velocities. In general, therefore, the present analysis is restricted to low subsonic Mach number flows having nonuniform temperature and density fields.

A second major difference between the present work and previous investigations is that the analysis herein is carried out in terms of the vortical, potential, and entropy modes of the shear layer. It is well known that a fluid motion may be described through interactions among these modes.[11] This type of analysis is beneficial because it provides insight into the physics of the problem. For example, in aeroacoustics,[12] such studies have served to clarify the distinction between the generated sound (a potential field), and the sound source (a vortical field). Flandro[13] has utilized such an approach in his study of vortex shedding mechanisms involved in solid propellant rocket motor instability.

In this paper, the inviscid spatial instability of a low Mach number shear layer with temperature and density gradients is analyzed by applying the Bernoulli enthalpy theory of aeroacoustics.[12,14] This results in an unambiguous separation of the acoustic, or potential, and vortical modes of the flow not obtained in earlier shear-layer instability studies. In what follows, a linearized three-dimensional analysis is first developed. With the help of a simple example it is then shown that the acoustic mode does not exhibit the exponential spatial growth characteristics of shear-layer instability. Hence, the shear-layer instability represents instabilities of the vortical and entropy modes. When these results are applied to a two-dimensional, parallel shear-layer, the vortical mode decouples from the entropy mode so that the shear-layer instability becomes equivalent to the instability of the vortical mode. Unstable vortical mode wave numbers are determined for the case of a hyperbolic tangent velocity profile for two different types of temperature distributions. These results show that depending upon the temperature profile, the degree of instability of the shear-layer can either increase or decrease compared to the constant temperature case. Finally, a solution is given for the acoustically excited component of the vortical mode for a simple case, and the coupling between the acoustic and vortical modes is discussed.

Formulation

Consider a flow described by a steady velocity $\bar{v}(x)$ and steady-state temperature and density distributions $\bar{T}(x)$ and $\bar{\rho}(x)$. The fluid is assumed to be a perfect gas with constant specific heats. For simplicity, viscous effects, heat conduction, and the presence of heat addition sources are neglected; however, an unsteady mass source located somewhere in the flow is included. There are two reasons for doing so. First, such a flow corresponds to a situation where the fluid is acoustically excited by external means. Second, it simplifies the interpretation of the instability equations, as will be evident shortly. Using these assumptions, the conservation equations expressed in terms of the enthalpy h and entropy s can be written in the following form[12]:

Continuity:

$$\frac{\partial h}{\partial h} + (v \cdot \nabla)\, h + a^2 \nabla \cdot v = m' \tag{1}$$

Momentum:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v + \nabla h = T \nabla s \tag{2}$$

Energy:

$$\frac{\partial s}{\partial t} + (v \cdot \nabla) s = 0 \tag{3}$$

Perturbations of the steady state are considered. The velocity perturbation may be written as the sum of a rotational, incompressible component and an irrotational component; that is,

$$v' = u' + \nabla\phi$$

where

$$\nabla \cdot u' = 0 \qquad \text{and} \qquad \nabla \times \nabla\phi = 0$$

It should be noted that the unsteady vorticity r' is given by the u' component alone as $r' = \nabla \times u'$.

In the spirit of instability analysis, the linearized equations for the perturbations are analyzed. Substituting the preceding expression for v' into the momentum equation and using standard vector identities, one obtains

$$\begin{aligned}\frac{\partial u'}{\partial t} &+ (\bar{v} \cdot \nabla)\, u' + (u' \cdot \nabla)\bar{v} + \nabla B'(T' \times \nabla\bar{s}) \\ &- \bar{T}\, \nabla s' = \nabla\phi \times (\nabla \times \bar{v})\end{aligned} \tag{4}$$

where

$$B' = h' + \frac{\partial\phi}{\partial t} + (\bar{v} \cdot \nabla)\, \phi$$

is a variable similar to the Bernoulli enthalpy considered in Ref. 14. Next, substituting the perfect gas relation $T' = h'/C_p$ into Eq. (4) yields

$$\begin{aligned}\frac{\partial u'}{\partial t} &+ (\bar{v} \cdot \nabla)\, u' + (u' \cdot \nabla)\bar{v} + \nabla B' - \frac{B'}{C_p}\nabla\bar{s} - \bar{T}\nabla s' \\ &= \nabla\phi \times \ (\nabla \times \bar{v}) - \frac{1}{C_p}\left[\frac{\partial\phi}{\partial t} + (\bar{v} \cdot \nabla\ \phi)\right]\nabla\bar{s}\end{aligned} \tag{5}$$

The continuity equation for the perturbations may be written as

$$\frac{\bar{D}^2\phi}{Dt^2} - a^2 \nabla^2 \phi - (\nabla\phi \cdot \nabla)\bar{h} = \frac{\bar{D}B'}{Dt} + (u' \cdot \nabla)\bar{h} + m' \quad (6)$$

and the entropy equation as

$$\frac{\bar{D}s'}{Dt} + (u' \cdot \nabla)\bar{s} = -(\nabla\phi \cdot \nabla)\bar{s} \quad (7)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\bar{v} \cdot \nabla)$$

Equations (5-7) represent a set of coupled equations for the vortical, potential, and entropy modes of the fluid motion. An equation of state, which will not be discussed here, may be used to complete the system. The nature of the preceding equations is discussed in detail in Ref. 12, where it is shown that B', u', and s' may be identified with the vortical and entropy modes. On the other hand, the velocity potential ϕ satisfies an acoustic wave-type equation and, thus, may be identified with the acoustic mode with $\nabla\phi$ being interpreted as the acoustic velocity.

The utility of the preceding formulation rests on two additional assumptions that appear reasonable (at least for low Mach numbers and low frequencies), although they cannot be proven rigorously. In what follows these assumptions are stated first, their implications are then outlined, and their use supported by use of a simple example.

The first assumption is that the acoustic mode does not exhibit the spatial growth that is characteristic of shear-layer instability. This implies that the instability is of the vortical and entropy modes alone. The second assumption is that the acoustic mode is decoupled from the vortical and entropy modes in the initial region of shear-layer formation where the instability occurs. This implies that sound is generated largely by fluid motion away from the initial shear-layer instability region. This assumption does not, of course, preclude feedback from the generated sound field on the initial shear layer development.

The justification for both of these assumptions lies in the widely disparate length scales of the acoustic, vortical, and entropy modes. The acoustic length scale is the acoustic wavelength which is based on the sound propagation speed according to the wave equation, Eq. (6). The vortical and entropy modes, which satisfy convective type equations, Eqs. [(5-7)], have length scales based on the flow velocity v. Thus, the ratio of the vortical and entropy length scales to the acoustic length scale is of the order of the flow Mach number, which is assumed to be small in this analysis.

These statements may be clarified by means of an example. Consider a duct which extends from $x = 0$ to $x = L$ and let the shear layer originate at a location $x = L_1$, between 0 and L downstream of a small obstruction. Let the flow Mach number be small and restrict attention to frequencies below cutoff of the transverse acoustic modes of the duct. Then a Helmholtz equation corresponding to Eq. (6) may be written as

$$\frac{d^2\Phi}{dx^2} + k^2\Phi = F \quad (8)$$

For simplicity, temperature gradients and the mass source m' may be ignored without affecting the arguments which follow.

The solution Φ may be required to satisfy the following general boundary conditions:

$$\frac{d\Phi}{dx} + R_1\Phi = 0 \qquad \text{at } x = 0$$

$$\frac{d\Phi}{dx} + R_2\Phi = 0 \qquad \text{at } x = L$$

where R_1 and R_2 are constants that may be real or complex.

The solution of Eq. (8) in terms of the relevant Green's function G is given by

$$\Phi(x) = \int_{x_0 = L_1}^{L} G(x; x_0)\, F(x_0)\, dx_0 \quad (9)$$

since it may be shown that F vanishes upstream of the shear layer origin.

It is readily shown that G is symmetric in x and x_0, that it varies on the scale of the relevant acoustic wavelength, and that it has no exponential growth characteristics. If the distance between L_1 and L is small compared to an acoustic wavelength, then Eq. (9) may be approximated by

$$\Phi(x) = G(x; \bar{x}) \int_{x_0 = L_1}^{L} F(x_0)\, dx_0 \sim AG(x; \bar{x})$$

where $\bar{x}$ is an x location between L_1 and L and A is constant. Thus, $\Phi(x)$ exhibits a spatial dependence similar to that of G, which is not exponential. Therefore, Φ does not exhibit the exponential growth characteristic of shear-layer instability.

If the distance between L_1 and L is comparable to or greater than the acoustic wavelength, the same argument is still applicable because $\Phi(x)$ can be expressed as follows:

$$\Phi(x) = \sum_{n=1}^{N} G(x; \bar{x}_n) \int_{x_n}^{x_{n+1}} F(x_0)\, dx_0$$

where $x_1 = L_1$, $x_N = L$, and the distance $x_{n+1} - x_n$ is small compared to an acoustic wavelength ($\bar{x}_n$ is now a typical value between x_n and x_{n+1}). The integrals are again constant and the variation of $\Phi(x)$ is similar to that of G.

The second assumption, that Φ is decoupled from the vortical and entropy modes in the initial region of shear-layer instability, is a consequence of the fact that Φ depends upon the integral of F over the entire extent of the shear layer. This suggests that the solution for Φ should only be weakly affected by the solutions for the vortical and entropy modes in the initial linear instability region of the shear layer which occupies only a small part of the total shear layer. This observation is supported by the success of classical aeroacoustics theories that do not specifically account for shear-layer instability in their description of turbulence-generated sound.

The preceding comments are now used to "interpret" Eqs. (5-7). First, consider the case when the unsteady source m' dominates the right-hand side of Eq. (6). In this case, the acoustic mode ϕ is decoupled from the vortical and entropy modes over the entire extent of the shear layer. Terms involving ϕ in Eqs. (5) and (7) may thus be evaluated and considered to be forcing terms for the vortical and entropy modes. Since these equations are linear, their solutions may be written in terms of complementary and particular solutions. The former describes the solution when the inhomogeneous parts of the equations, involving the ϕ terms, are suppressed. Furthermore, the vortical and entropy modes tend to zero in the region outside the shear layer, where the temperature and velocity gradients are negligible. When applied to a sheared parallel flow (as is done in the next section), this implies that the homogeneous equations for the vortical and entropy modes also satisfy homogeneous boundary conditions. The solutions to these equations are, therefore, eigensolutions and may be interpreted (as is normally done)[15,16] as describing the natural instability of the shear layer.

Next, consider the case when the external acoustic excitation m' is zero. Then, according to the second assumption, the acoustic waves are largely generated by fluid motion away from the shear layer's linear instability region. Thus, the right-hand side of Eq. (6) may be neglected in the initial instability region so that ϕ is decoupled from the vortical and entropy

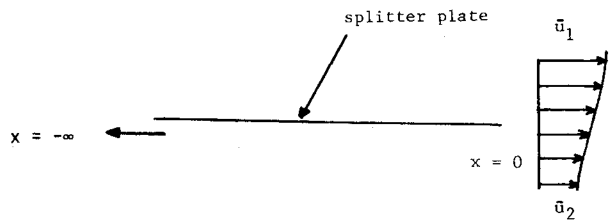

Fig. 1 Two-dimensional parallel shear layer formed downstream of a splitter plate.

modes in this region. Hence, again the ϕ terms may be taken to be forcing terms for the vortical and entropy modes in Eqs. (5) and (7) in the linear instability region of the shear layer. Therefore, as in the case when m' dominates the right-hand side of Eq. (6), the homogeneous solutions of Eqs. (5) and (7) may be taken to describe the natural instability of the shear layer.

Two-Dimensional Parallel Shear Layer

As an example, consider a two-dimensional parallel shear layer formed downstream of a splitter plate as shown in Fig. 1 for which

$$\bar{v} = \bar{u}(y)\boldsymbol{i}, \qquad \bar{T} = \bar{T}(y), \qquad \bar{p} = \text{const}$$

The preceding implies that for a perfect gas with constant specific heats[17]

$$\frac{1}{C_p}\frac{d\bar{s}}{dy} = \frac{1}{\bar{T}}\frac{d\bar{T}}{dy}$$

First, the equation for the vorticity perturbation $r'\boldsymbol{k}$ is obtained by assuming harmonic time dependence, taking the curl of Eq. (5) and applying the result to a two-dimensional flow, yielding

$$i\omega r + \bar{u}\frac{\partial r}{\partial x} - u_2\frac{d^2\bar{u}}{dy^2} - \frac{1}{\bar{T}}\frac{d\bar{T}}{dy}\left(\frac{\partial B}{\partial x} - \bar{T}\frac{\partial S}{\partial x}\right) = -F_1(\Phi) \tag{10}$$

where

$$F_1(\Phi) = -\frac{\partial \Phi}{\partial y}\frac{d^2\bar{u}}{dy^2} - \left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right)\frac{d\bar{u}}{dy} + \frac{1}{\bar{T}}\frac{d\bar{T}}{dy}\left(i\omega\frac{\partial\Phi}{\partial x} + \bar{u}\frac{\partial^2\Phi}{\partial x^2}\right)$$

Using the x component of Eq. (5), B and S may be eliminated from Eq. (10) to get

$$i\omega r + \bar{u}\frac{\partial r}{\partial x} - \bar{u}_2\frac{d^2\bar{u}}{dy^2} + \frac{1}{\bar{T}}\frac{d\bar{T}}{dy} \times \left(i\omega u_1 + \bar{u}\frac{\partial u_1}{\partial x} + u_2\frac{d\bar{u}}{dy}\right) = -F_2(\Phi) \tag{11}$$

where u_1 and u_2 are the axial and normal velocity components and

$$F_2(\Phi) = F_1(\Phi) + \frac{1}{\bar{T}}\frac{d\bar{T}}{dy}\frac{d\bar{u}}{dy}\frac{\partial\Phi}{\partial y}$$

Thus, the vortical mode, now characterized by r, u_1, and u_2, has been decoupled from the entropy mode characterized by S.

Noting that $\nabla\cdot u' = 0$, a stream function ψ defined by

$$u_1 = -\frac{\partial\psi}{\partial y} \qquad \text{and} \qquad u_2 = \frac{\partial\psi}{\partial x}$$

is introduced and substituted into Eq. (11) to yield

$$\left[i\omega + \bar{u}\frac{\partial}{\partial x}\right]\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{1}{\bar{T}}\frac{d\bar{T}}{dy}\frac{\partial x}{\partial y}\right] - \frac{\partial\psi}{\partial x} \times \left[\frac{d^2\bar{u}}{dy^2} - \frac{1}{\bar{T}}\frac{d\bar{T}}{dy}\frac{d\bar{u}}{dy}\right] = -F_2(\Phi) \tag{12}$$

A solution for ψ consisting of a complementary part ψ_1 and a particular part ψ_2 will be sought. The homogenous solution ψ_1 is written in the form

$$\psi_1 = CA_1(y)e^{-i\alpha x}$$

Substituting this expression into Eq. (12) and suppressing the right-hand side yields the following equation for $A_1(y)$

$$\frac{d}{dy}\left(\frac{1}{\bar{T}}\frac{dA_1}{dy}\right) + \left[\left(\frac{\alpha}{\omega - \alpha\bar{u}}\right)\frac{d}{dy}\left(\frac{1}{\bar{T}}\frac{d\bar{u}}{dy}\right) - \frac{\alpha^2}{\bar{T}}\right]A_1 = 0 \tag{13}$$

The boundary conditions for A_1 are obtained by noting that the vortical velocities vanish[12] in regions where the temperature and velocity gradients vanish. Thus, for a free shear layer for which the velocity and temperature gradients tend to zero at infinity

$$A_1 \rightarrow 0 \qquad \text{as} \qquad y \rightarrow \pm\infty$$

The wave number α is an eigenvalue for the homogeneous problem, and $A_1(y)$ is the corresponding eigenfunction. The constant C is to be chosen in a way that will require the combined solution for ψ (i.e., $\psi_2 + \psi_1$) to satisfy an appropriate initial condition at the shear-layer origin. This aspect is discussed further shortly. If the imaginary part of α is positive [i.e., Im $(\alpha) > 0$], then ψ_1 describes a spatially growing vortical wave. The phase velocity of the wave is given by $c_{ph} = \omega/\text{Re}(\alpha)$.

Note that Eq. (13) reduces to the Rayleigh instability equation[15] when steady-state temperature gradients are absent. However, when temperature gradients are present, the quantity $\{d/dy\,[(1/\bar{T})(d\bar{u}/dy)]\}$ plays the same role as does the quantity $d^2\bar{u}/dy^2$ in a purely incompressible flow. The location in the shear layer where this quantity vanishes is termed the generalized inflection point. It is possible, depending on the velocity and temperature profiles, that more than one such point may exist in the shear layer. The significance of this occurrence is discussed in the literature.[18] In this paper, cases in which only one generalized inflection point occurs are considered. It should also be pointed out that for a neutral disturbance [i.e., one for which Im $(\alpha) = 0$], the phase velocity of the wave coincides with the flow velocity at the generalized inflection point.

The velocity and temperature profiles considered here as examples are shown in Figs. 2–4 in nondimensional form. The normalizing length scale for both the velocity and temperature profiles is taken to be Θ, the momentum thickness of the shear layer. The normalizing velocity and temperature are the maximum steady-state values attained in the flow. The assumed hyperbolic tangent velocity profile takes the form

$$\frac{\bar{u}}{\bar{u}(+\infty)} = 1/2\,[1 + \tanh(y/\Theta)] \tag{14}$$

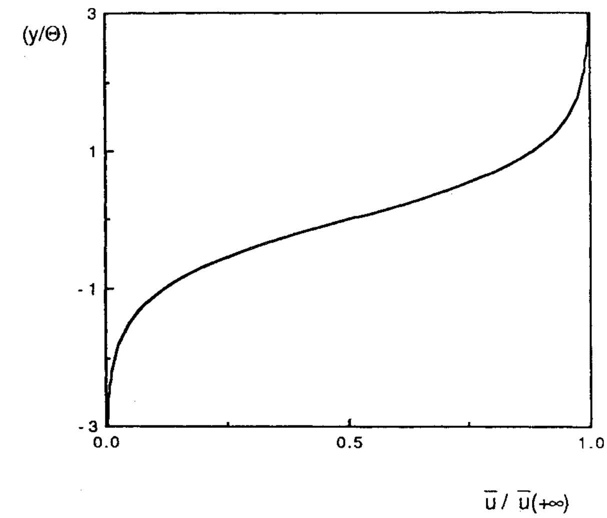

Fig. 2 Hyperbolic tangent velocity profile defined by Eq. (14)

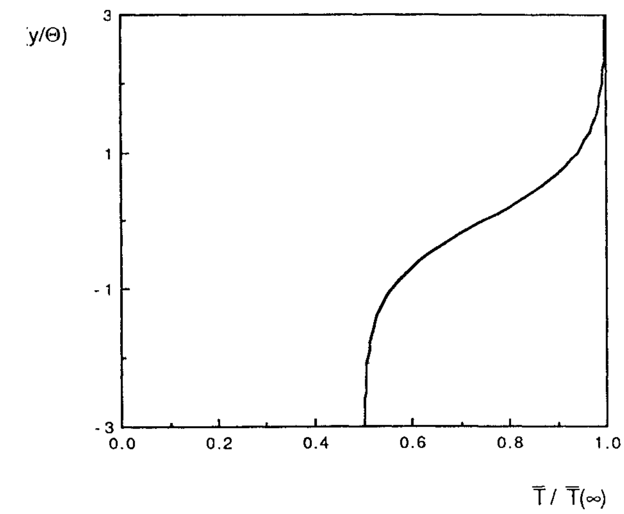

Fig. 4 Temperature profile defined by Eq. (16)

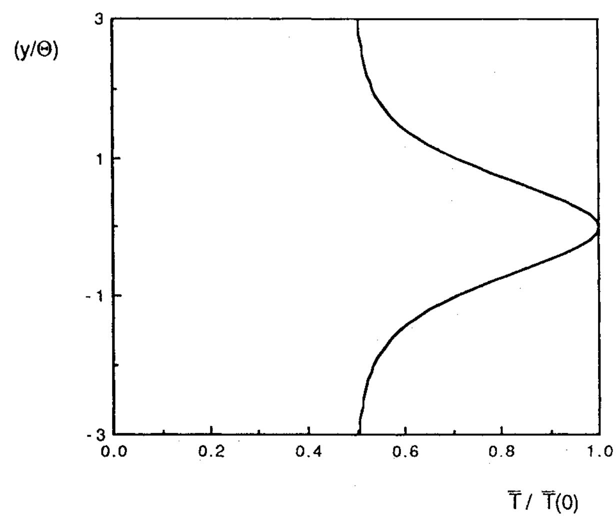

Fig. 3 Temperature profile defined by Eq. (15)

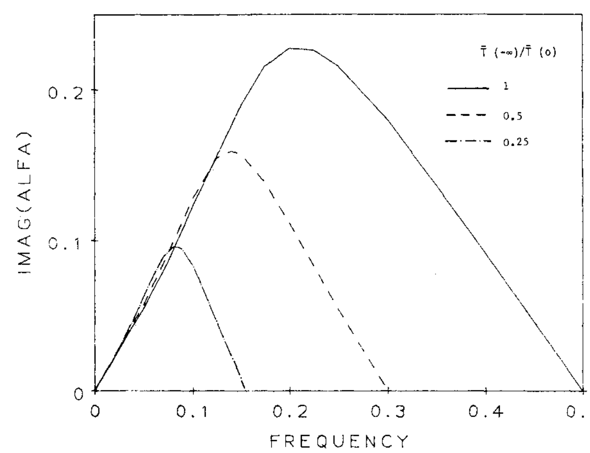

Fig. 5 The Im($\alpha\Theta$) as a function of nondimensional frequency for the velocity and temperature profiles of Eqs. (14) and (15).

The two considered temperature profiles are

$$\frac{\bar{T}}{\bar{T}(0)}=\frac{\bar{T}(-\infty)}{\bar{T}(0)}+\left[1-\frac{\bar{T}(-\infty)}{\bar{T}(0)}\right]\operatorname{sech}^2(y/\Theta) \tag{15}$$

and

$$\frac{\bar{T}}{\bar{T}(\infty)}=\frac{1}{2}\,[1+\tanh\,(y/\Theta)]+\frac{1}{2}\,\frac{\bar{T}(-\infty)}{\bar{T}(\infty)}\,[1-\tanh(y/\Theta)] \tag{16}$$

Some comments are in order regarding the choice of the assumed temperature profiles. Equation (15) represents a temperature distribution similar to one that would be obtained with a diffusion flame located at $y=0$. Such a situation may arise, for example, when oxidizer flows in the region $0 < y < +\infty$ and fuel flows in the region $-\infty < y < 0$. Equation (16) represents a temperature profile similar to the velocity profile and may be thought of as the temperature distribution obtained when coflowing streams at temperatures $\bar{T}(-\infty)$ and $\bar{T}(\infty)$ mix. Similar temperature profiles have also been considered by other investigators.[19,20]

Using these velocity and temperature distributions, Eq. (13) has been solved numerically to determine the eigenvalue α, using techniques suggested in Ref. 16. The results are shown in Figs. 5 and 6. Figure 5 plots Im($\alpha\Theta$) as a function of the nondimensional frequency for different values of $\bar{T}(-\infty)/\bar{T}(0)$ for the temperature profile of Eq. (15). Figure 6 plots the same for different values of $\bar{T}(-\infty)/\bar{T}(\infty)$ for the temperature profile of Eq. (16). In both cases, the results for $\bar{T}(-\infty)/\bar{T}(0)=\bar{T}(-\infty)/\bar{T}(\infty)=1$ correspond to a constant temperature shear layer, which has been discussed in detail in the literature.[21] For this situation, the vortical mode is unconditionally unstable for nondimensional frequencies in the range from 0 to 0.5. The frequency at which Im($\alpha\Theta$) is maximum is the most unstable frequency.

The calculations show that for both temperature profiles the frequency range of the instability is reduced as the temperature ratios $\bar{T}(-\infty)/\bar{T}(0)$ or $\bar{T}(-\infty)/\bar{T}(\infty)$ are reduced. However, for the profile of Eq. (15) the growth rate is lower than that in the constant temperature case, whereas for the profile of Eq.(16) it is larger. This difference in behavior illustrates the importance of the structure of the temperature profile in determining the shear-layer instability characteristics. In connection with the results obtained using Eq. (15), it may be noted that computations[22,23] as well as experiments[24] with flows having diffusion-limited reactions show qualitatively similar trends. That is, with increase in the value of the heat release [which would decrease the parameter $\bar{T}(-\infty)/\bar{T}(0)$] the shear layer exhibits decreased growth rates as well as a reduction in the values of the most unstable frequency.

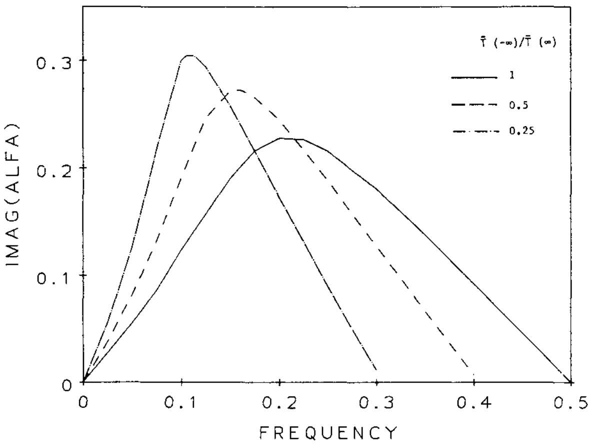

Fig. 6 The Im($\alpha\Theta$) as a function of nondimensional frequency for the velocity and temperature profiles of Eqs. (14) and (16).

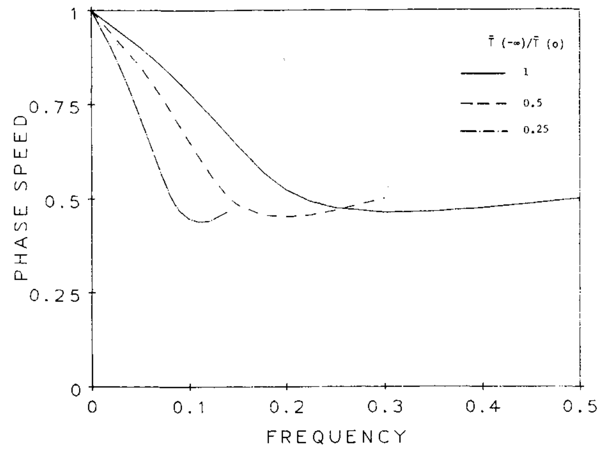

Fig. 7 Phase velocity as a function of frequency for the velocity and temperature profiles of Eqs. (14) and (15)

Figures 7 and 8 show the calculated phase velocities of the unstable vortical modes as a function of the frequency for the temperature distributions of Eqs. (15) and (16), respectively. As expected, the phase velocities of the neutral waves agree with the steady-state velocities at the corresponding generalized inflection point. For the set of profiles corresponding to Eqs. (14) and (15), the inflection point y_c is located at $y_c = 0$, and for the profiles of Eqs. (14) and (16) its location is given by

$$\tanh(y_c/\Theta) = -\left[\frac{1+\bar{T}(-\infty)/\bar{T}(\infty)}{1-\bar{T}(-\infty)/\bar{T}(\infty)}\right] + \left\{\left[\frac{1+\bar{T}(-\infty)/\bar{T}(\infty)}{1-\bar{T}(-\infty)/\bar{T}(\infty)}\right]^2 - 1\right\}^{0.5} \tag{17}$$

Next, the acoustically forced problem is considered. The particular solution for Eq. (12) is now obtained for a simple case where $M^2 \ll 1$ and the temperature gradients are also small. It is further assumed that the sound generated by the external driving dominates the sound generated by fluid motion. For this case, a plane wave solution for Φ satisfying Eq. (6) has, approximately, the form

$$\Phi = \hat{\Phi}\, e^{-ikx_0}\, e^{i(\omega t - kx)}$$

where the factor e^{-ikx_0} accounts for the phase of the wave. Using the preceding expression for Φ, F_2 (Φ) becomes

$$F_2\,(\Phi) = k^2\left[\frac{\mathrm{d}\bar{u}}{\mathrm{d}y} + \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}y}\left(\frac{\omega}{k} - \bar{u}\right)\right]\hat{\Phi} e^{-ikx_0}$$

Expressing ψ_2 as

$$\psi_2 = \hat{\psi}_2\,(y)\, e^{i(\omega t - kx)}$$

and substituting this expression into Eq. (12), the following equation is obtained for $\hat{\psi}_2$:

$$\frac{\mathrm{d}^2\hat{\psi}_2}{\mathrm{d}y^2} - \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}y}\frac{\mathrm{d}\hat{\psi}_2}{\mathrm{d}y} + \left[\left(\frac{k}{\omega - k\bar{u}}\right)\left(\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2} - \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}y}\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}\right) - k^2\right]\hat{\psi}_2$$

$$= \frac{ik^2}{(\omega - k\bar{u})}\left[\frac{\mathrm{d}\bar{u}}{\mathrm{d}y} + \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}y}\left(\frac{\omega}{k} - \bar{u}\right)\right]\hat{\Phi}\, e^{-ikx_0} \tag{18}$$

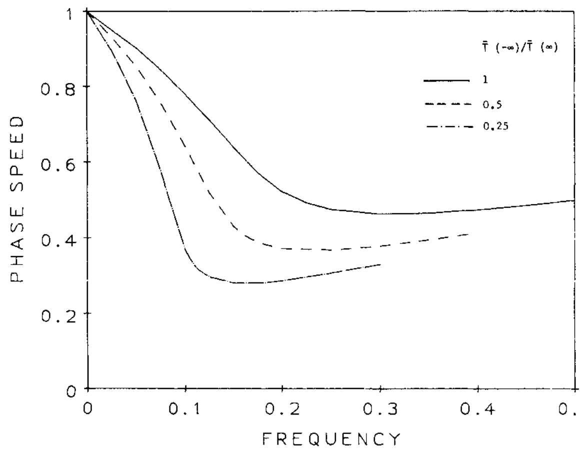

Fig. 8 Phase velocity as a function of frequency for the velocity and temperature profiles of Eqs. (14) and (16)

It is convenient now to introduce the following nondimensional quantities

$$u^* = \frac{u}{a_r}, \quad y^* = \frac{y}{\lambda}, \quad \omega^* = \frac{\omega\lambda}{a_r}, \quad T^* = \frac{T}{\bar{T}_m}, \quad x^* = \frac{x}{\lambda}$$

$$k^* = k\lambda, \quad \hat{\psi}_2^* = \frac{\hat{\psi}_2}{\bar{u}_m\lambda}, \quad \hat{\Phi}^* = \frac{\hat{\Phi} e^{-ik^*x_0^*}}{a_r\lambda}, \quad M_r^* = \frac{\bar{u}_m}{a_r}, \quad M^* = \frac{\bar{u}}{a_r}$$

where $\bar{u}_m$ and $\bar{T}_m$ are the maximum steady-state velocity and temperature, respectively. Substituting these nondimensional quantities into Eq. (18) yields the following equation:

$$\frac{\mathrm{d}^2\hat{\psi}_2}{\mathrm{d}y^2} - \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}y}\frac{\mathrm{d}\hat{\psi}_2}{\mathrm{d}y} + \left[\left(\frac{k}{\omega - kM}\right)\left(\frac{\mathrm{d}^2M}{\mathrm{d}y^2} - \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}y}\frac{\mathrm{d}M}{\mathrm{d}y}\right) - k^2\right]\hat{\psi}_2$$

$$= \left(\frac{ik}{\omega - kM}\right)\left[\frac{1}{M_r}\frac{\mathrm{d}M}{\mathrm{d}y} - \frac{1}{M_r\bar{T}}\left(\frac{\omega}{k} - M\right)\right]\hat{\Phi} \tag{19}$$

where the * superscript has been omitted, as all quantities are now nondimensional.

With the assumed form for Φ, the chosen nondimensionalization scheme, and the assumptions of small Mach number and temperature variations, the following relations hold:

$$\frac{k}{\omega}=\frac{\omega}{k}=1,\ kM \ll \omega$$

$$\frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}y} \sim 0\left(\frac{\mathrm{d}M}{\mathrm{d}y}\right)$$

Then, in the limit when $M^2 \ll (\Theta/\lambda)^2$, Eq. (19) reduces to

$$\frac{\mathrm{d}^2\hat{\psi}_2}{\mathrm{d}y^2}-k^2\,\hat{\psi}_2=i\left[\frac{1}{M_r}\frac{\mathrm{d}M}{\mathrm{d}y}-\frac{1}{M_r\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}y}\right]\hat{\Phi} \tag{20}$$

and $\hat{\psi}_2$ must satisfy the following boundary conditions

$$\hat{\psi}_2\,(-\infty)=\hat{\psi}_2\,(\infty)=0$$

Using Green's function techniques, the following solution for $\hat{\psi}_2$ is obtained:

$$\hat{\psi}_2=-\frac{i\hat{\Phi}}{M_r}\int_{-\infty}^{+\infty} G(y;\,y_0)\left(\frac{\mathrm{d}M}{\mathrm{d}y_0}-\frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}y_0}\right)\mathrm{d}y_0 \tag{21}$$

where G, the relevant Green's Function for the problem, is given by

$$\begin{aligned} G(y;\,y_0)&=\frac{1}{2k}\,e^{ky}\,e^{-ky_0} \qquad -\infty < y < y_0 < \infty \\ &=\frac{1}{2k}\,e^{-ky}\,e^{-ky_0} \qquad -\infty < y_0 < y < \infty \end{aligned} \tag{22}$$

It should be noted that the length scale over which variations in $\hat{\psi}_2$ occur coincides with the length scale of G which, according to Eq. (22), is $y=1/k$; that is, $\hat{\psi}_2$ varies on the scale of the acoustic wavelength λ. This is to be contrasted with the momentum thickness of the shear layer, which is the length scale of variation in y for the homogeneous problem. Thus, $\hat{\psi}_2$ is basically constant in the region of intense shear, whereas $\hat{\psi}_1$ varies over it as long as low frequencies (for which $\lambda \gg \Theta$) are considered.

Next, the coupling constant C [see definition of ψ_1 below Eq. (12)] will be evaluated. The dimensional form ψ_1^* is

$$\psi_1^*=C\,A_1^*\,(y)\,e^{-i\alpha^*x^*}$$

This is nondimensionalized as follows

$$\psi_1^*=\frac{\psi_1}{\bar{u}_m\lambda} \qquad \text{with } \alpha^*=\alpha\lambda \qquad \text{and } x^*=\frac{x}{\lambda}$$

The solution for the total stream function ψ is given in nondimensional form by $\psi=\psi_1+\psi_2$, where ψ_2 is the particular solution solved for as in the preceding example. The constant C, which describes the influence of the acoustics on the unstable vortical wave ψ_1, is to be determined by prescribing a suitable initial condition that ψ must satisfy.

Consider the situation depicted in Fig. 1. A thin splitter plate extending from $x=-\infty$ to $x=0$ separates two streams of different velocities. These streams merge downstream of the splitter plate trailing edge to form the shear layer of interest. For simplicity, it will be assumed that both streams are at the same constant steady-state temperature. If each of the two streams has a reasonably uniform velocity, it is clear that the trailing edge of the splitter plate coincides with the initial inflection point (i.e., layer of maximum mean shear) of the shear layer.

The condition to be specified at the trailing edge on the unsteady motion is debatable. For example, Flandro,[13] in his work on vortex driving mechanisms in solid propellant rocket motor combustion instabilities, suggests that the unsteady vorticity be set to zero at the trailing edge. However, it is readily shown for the case of a constant temperature, two-dimensional parallel shear layer that the unsteady vorticity obtained from the shear-layer instability solutions is identically zero by itself at the inflection point. This is clear from Eq. (12), where at the inflection point

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}=0$$

and the right-hand side is to be suppressed for the instability solutions. Then, for no temperature gradients it follows that $\nabla^2\psi_1$, which represents the unstable unsteady vorticity, is identically zero at this location. Thus, setting the overall unsteady vorticity at the inflection point to zero is identical to setting the acoustically driven vorticity to zero at this location. This procedure will not yield the coupling constant C which would remain indeterminate.

One approximation to the flow at the trailing edge is obtained by requiring the dividing streamline to be parallel to the trailing edge at all times. This condition may be satisfied by requiring the transverse velocity fluctuations to vanish at this location. Since the acoustic motions associated with the plane wave are purely axial, the transverse vortical velocity fluctuations must vanish at the trailing edge. Thus, at the trailing edge of the splitter plate, which is located at $x=0$ and $y=y_c$, the following initial condition must be satisfied:

$$\left[\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial x}\right]_{\substack{x=0\\ y=y_c}}=0 \tag{23}$$

Now, at the trailing edge

$$\left[\frac{\partial\psi_1}{\partial x}\right]_{\substack{x=0\\ y=y_c}}=-i\partial CA_1(y_c)\,e^{i\omega t}$$

Since $A_1(y_c)$ may be set equal to $1+i0$ without any loss of generality, then

$$\left[\frac{\partial\psi_1}{\partial x}\right]_{\substack{x=0\\ y=y_c}}=-i\alpha Ce^{i\omega t} \tag{24}$$

Using the preceding expression for ψ_2, one gets

$$\left[\frac{\partial\psi_1}{\partial x}\right]_{\substack{x=0\\ y=y_c}}=ik\,\hat{\psi}_2(y_c)\,e^{i\omega t} \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (23) and solving for C, one obtains

$$C=\frac{k\,\hat{\psi}_2(y_c)}{\partial} \tag{26}$$

The quantity $\hat{\psi}_2\,(y_c)$ is readily evaluated at low frequencies in the low Mach number limit $[M^2 \ll (\Theta/\lambda)^2]$ irrespective of the velocity profile from Eq. (21) (in nondimensional form), by noting that the Green's function G varies on the scale of the acoustic wavelength λ, whereas $\mathrm{d}M/\mathrm{d}y$ has a length scale of the order of the momentum thickness Θ of the shear layer. Thus, for low frequencies, where the relation $\lambda/\Theta \gg 1$ holds, G is basically constant over the region of variation of M (or dM/dy). Then, setting $y_c = 0$ for convenience, it may be shown from Eqs. (21) and (22) that

$$\hat{\psi}_2(0) = \frac{-i\hat{\Phi}}{2kM_r}\Delta M \tag{27}$$

where ΔM is the difference between the Mach numbers at $y \to +\infty$ and $y \to -\infty$. Substituting Eq. (27) into Eq. (26), one obtains

$$C = \frac{-i\hat{\Phi}\Delta M}{2M_r\,\partial} \tag{28}$$

The magnitude and phase of the unstable vortical wave with respect to the acoustic motions is thus established.

The matching also indicates that the vortical motions due to ψ_1 and ψ_2 are of the same order at the shear-layer origin. However, the motions associated with ψ_2 do not exhibit spatial growth like the motions associated with ψ_1. Hence, downstream of the shear-layer origin the unstable vortical motions due to ψ_1 will tend to dominate the motions due to ψ_2.

Summary and Conclusions

The vortical mode instability of low Mach number parallel shear layers with temperature and density gradients has been studied. It has been demonstrated that, depending on the steady-state temperature profile in the shear layer, the growth rates of the unstable vortical modes can either increase or decrease as compared to the constant temperature case. The influence of the acoustic mode on the vortical mode has been shown to be twofold. First, in the low Mach number limit the acoustic mode causes vortical motions that have a length scale of the order of the acoustic wavelength. These vortical motions correspond to the particular solution of the inhomogeneous vorticity equation. Second, and more importantly, the acoustic mode sets the initial value of the unstable vortical modes corresponding to the solutions of the homogeneous vorticity equation.

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